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Ising Model with Restricted Interactions on General Cayley Tree of Arbitrary Order

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Abstract: We study an Ising model on a general Cayley tree with competing interaction of next-nearest-neighbour of two types; prolonged and one-level *k*-tuple interactions. Vannimenus covers a study of Ising model on competing nearest-neighbour interaction J_1 and prolonged next-nearest-neighbour interaction J_p. Later Mariz et al extend Vannimenus work by adding a one-level interaction J_o for order 2 with the existence of external magnetic field. And recently Ganikhodjaev et al found only simple modulated phases exist for Ising model with second neighbour prolonged and one-level competing interaction of order 2. So, in this paper, we study the Ising model with competing interaction $J_0 \neq 0$, $J_p \neq 0$ and $J_1 = 0$ to identify the preservation of those phases for higher order *k*. We derived two types of general system of equations and describe the phase diagram for both cases of even and odd order.

Key words: Ising models, Cayley tree, prolonged next-nearest-neighbour, one-level next-nearest-neighbour.

INTRODUCTION

Since the time of Gibbs, the field of statistical mechanics has been given a lot of attention in physics world. One approach in statistical mechanics known as the model-building approach is an attempts to gain insight into real situation by analyzing on simple models. Among several models exist in statistical mechanics, an Ising model is the most simplified model and has been studies extensively by the physicist. The model consists of a lattice of ‘spin’ variables which may take on only ±1 values.

Competing interaction of Ising model on a Cayley tree is equivalent to the standard Bethe-Peierls theory and is well-studied due to the appearance of non-trivial magnetic orderings (Mariz *et al.,* 1985). It is a suitable model to describe the phenomena of ferromagnetism in the physical substance (Kerson Huang, 1987).

The competing interaction of nearest-neighbor(NN) J_1 and prolonged next-nearest-neighbor(NNN) J_p has been considered by Vannimenus in (J. Vannimenus, 1981). He found a new modulated phase in addition to the expected paramagnetic and ferromagnetic phase and also found the multicritical Lifshitz point at zero temperature. Vannimenus model was then been extended by (Mariz *et al.,* 1985). Mariz et al covered the model with competing nearest-neighbour and next-nearest-neighbour interaction of two types; prolonged NNN– where the neighbourhood is at different level, and one-level NNN– where the neighbourhood is in the same level, in addition to the external magnetic field.

Later, CR da Silva et al (Da Silva C.R., and Coutinho S., 1985) studied the case competing interaction of first-, second- and third- neighbouring spin in the presence of external magnetic field. Only recently, Ganikhodjaev et al (J. Vannimenus, 1981) generalized the Ising model introduced by Vannimenus (J. Vannimenus, 1981) and (Simon, B., 1993) but without external magnetic field for arbitrary order *k*.

In addition, Ising model on Cayley tree of order two with competing prolonged and one-level *k*-tuple next-nearest-neighbour interactions has been discuss by Ganikhodjaev and Siti Fatimah in (Ganikhodjaev N., Siti Fatimah Zakaria, 2011). Only four types of phases including ferromagnetic and paramagnetic phase with simple modulated structure of antiphase and antiferromagnetic are found in their paper (Ganikhodjaev N., Siti Fatimah Zakaria, 2011). A question arises about the existence of modulated phase for larger order *k*.

The interest is to know whether the four phases in (Ganikhodjaev N., Siti Fatimah Zakaria, 2011) will be preserve or there will be arise new modelated phase for arbitrary order *k*. For this reason, the current interest in this paper is to generalize the Ising model on Cayley tree of arbitrary order with the existence only the competing interaction of next-nearest-neighbour interaction prolonged and one-level *k*-tuple. Then, we describe and discuss the difference of the phase diagram for higher order *k*; when *k* is even and *k* is odd.

Defination of Model:

Cayley Tree. A Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k + 1$ edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. Two vertices x and y; $x, y \in V$ are called nearest- neighbours if there exists an edge

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$l \in \Lambda$ connecting them, which is denoted by $l = \langle x, y \rangle$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree Γ^k, is the number of edges in the shortest path from x to y.
For a fixed $x^0 \in V$ we set
$W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \{x \in V \mid d(x, x^0) \leq n\}$
and L_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called the next-nearest-neighbours if $d(x, y) = 2$.

The next-nearest-neighbour vertices x and y are called one-level next-nearest-neighbour if $x, y \in W_n$ for some n and is denoted by $>\overline{x, y}<$. The next-nearest-neighbour vertices x, y that are not one-level are called prolonged next-nearest-neighbour vertices and is denoted by $>\widetilde{x, y}<$ (see Figure 1) .

The set of the direct successors of x is denoted by S(x) i.e., if $x \in W_n$ then $S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \ldots, k\}$. The collection S(x), will be called one-level k-tuple of neighbours. Note that if k = 2, then S(x) is a one-level next-nearest neighbours.

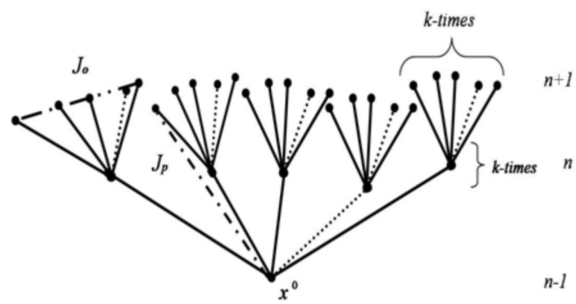


Fig. 1: Three successive generation of a semi-infinite Cayley tree (solid line: nearest neighbour interactions; dot-dashed line: next-nearest neighbour prolonged interactions; dot-dot-dashed line: next-nearest neighbour one-level interactions).

The model. Consider a semi-infinite Cayley tree Γ_+^k of *k*-th order i.e. an infinite graph without cycles, with exactly *k* + 1 edges issuing from each vertex, except a root $x^0 \in V$ which only emanates *k* edges from the vertex. The model with competing prolonged and one-level *k*-tuple next-nearest neighbour interactions has the Hamiltonian *H* as the following form:

$$H(\sigma) = -J_p \sum_{>\widetilde{x, y}<} \sigma(x)\sigma(y) - J_o \sum_{x \in V} \prod_{y \in S(x)} \sigma(y) \tag{1}$$

where first summation is over prolonged next-nearest-neighbours and second is over one-level *k*-tuple neighbours and $\sigma : V \to \{-1, 1\}$ a configuration on *V*.

1. General System:

In this paper, we extend the work of Ganikhodjaev and Siti Fatimah [3] in the general form. We produced a general recurrent system of equations with respect to semi-infinite Cayley tree Γ_+^k of order *k*.

We take into account the partition functions for all possible configurations in 2 successive generations (J. Vannimenus, 1981). Let $\overline{\sigma}_n : V \setminus V_n \to \{-1, 1\}$ be a fixed boundary condition and $\sigma_n : V_n \to \{-1, 1\}$ be an arbitrary configuration on V_n. Then one can compute energy of configuration σ_n with fixed boundary condition $\overline{\sigma}_n$ as follows:

$$H(\sigma_n \mid \overline{\sigma}_n) = -J_p \sum_{\substack{>\widetilde{x,y}< \\ x, y \in V_n}} \sigma_n(x)\sigma_n(y) - J_o \sum_{x \in V_{n-1}} \prod_{y \in S(x)} \sigma_n(y)$$
$$- J_p \sum_{\substack{>\widetilde{x,y}< \\ x \in V_{n-1}, \\ y \in V_{n+1}}} \sigma_n(x)\overline{\sigma}_n(y) - J_p \sum_{\substack{>\widetilde{x,y}< \\ x \in V_n, \\ y \in V_{n+2}}} \sigma_n(x)\overline{\sigma}_n(y) - J_o \sum_{x \in V_n} \prod_{y \in S(x)} \overline{\sigma}_n(y)$$

We define a partition functions with fixed boundary conditions:

$$Z^{(n)}(\overline{\sigma}_n) = \sum_{\sigma_n \in \Omega(V_n)} \exp(-\beta H(\sigma_n \mid \overline{\sigma}_n))$$

as a standard partition function for all configurations and

$$Z^{(n)}(\sigma(x^0), \sigma(y_i) \mid \overline{\sigma}_n) = \sum_{\substack{\sigma_n \in \Omega(V_n): \\ \sigma_n(x^0) = \sigma(x^0) \\ \sigma_n(y_i) = \sigma(y_i)}} \exp(-\beta H(\sigma_n \mid \overline{\sigma}_n))$$

is a partition function on V_n with fixed ($\sigma(x^0), \sigma(y_i)$) configuration. Also,

$$\begin{aligned} Z^{(n)}(\sigma_S \mid \overline{\sigma}_n) &= \sum_{\substack{\sigma_n \in \Omega(V_n): \\ \sigma_n = \sigma_S|_{V_1}}} \exp(-\beta H(\sigma_n \mid \overline{\sigma}_n)) \\ &= Z^{(n)}\begin{pmatrix} \sigma(y_1), \sigma(y_2), \ldots, \sigma(y_k) \\ \sigma(x^0) \end{pmatrix} \\ &= \exp\left(\frac{J_0}{T}\prod_{i=1}^{k}\sigma(y_i)\right) Z^{(n)}(\sigma(x^0), \sigma(y_i) \mid \overline{\sigma}_n) \end{aligned}$$

and

$$Z^{(n)} = \sum_{\sigma_S \in \Omega(S)} Z^{(n)}(\sigma_S \mid \overline{\sigma}_n)$$

where σ_S is a configuration $\sigma_S : x \cup S(x^0) \to \{-1,1\}$ and $\Omega(S)$ is the set of all configurations on $x \cup S(x^0)$.

We let a and b be parameter in the recurrent equations as

$$a = \exp\left(\frac{J_0}{kT}\right) \quad \text{and} \quad b = \exp\left(\frac{J_p}{T}\right) \quad .$$

It is reasonable to pay attention to only the independent variable of the partition function. In general, there are only 4 independent variables need to be considered. Now, for simplicity we let m be number of spins down i.e. $\sigma(y_i) = -1$ on W_1 level where $0 \le m \le k$.

A. Even k

The independent variables are as below:

$$z_1 = Z^{(n)}\begin{pmatrix} + + \ldots + \\ + \end{pmatrix} = a^k \left[Z^{(n)}(+,+) \right]^k$$

$$z_2 = Z^{(n)}\begin{pmatrix} - - \ldots - \\ + \end{pmatrix} = a^k \left[Z^{(n)}(+,-) \right]^k$$

$$z_3 = Z^{(n)}\begin{pmatrix} + + \ldots + \\ - \end{pmatrix} = a^k \left[Z^{(n)}(-,+) \right]^k$$

$$z_4 = Z^{(n)}\begin{pmatrix} - - \ldots - \\ - \end{pmatrix} = a^k \left[Z^{(n)}(-,-) \right]^k .$$

When the number of spins down m is even, then,

$$Z^{(n)}\begin{pmatrix}\sigma(y_1),\sigma(y_2),\dots,\sigma(y_k)\\ +\end{pmatrix} = z_1^{\frac{k-m}{k}} z_2^{\frac{m}{k}},$$
$$Z^{(n)}\begin{pmatrix}\sigma(y_1),\sigma(y_2),\dots,\sigma(y_k)\\ -\end{pmatrix} = z_3^{\frac{k-m}{k}} z_4^{\frac{m}{k}}.$$

And when the number of spins down m is odd, then,

$$Z^{(n)}\begin{pmatrix}\sigma(y_1),\sigma(y_2),\dots,\sigma(y_k)\\ +\end{pmatrix} = a^{-2k} z_1^{\frac{k-m}{k}} z_2^{\frac{m}{k}},$$
$$Z^{(n)}\begin{pmatrix}\sigma(y_1),\sigma(y_2),\dots,\sigma(y_k)\\ -\end{pmatrix} = a^{-2k} z_3^{\frac{k-m}{k}} z_4^{\frac{m}{k}} \quad .$$

We introduced a new variable u_i with $u_i = \sqrt[k]{z_i}$; $z_i = u_i^k$. Then the recurrence equations are reduced to these recurrent systems:

$$u_1' = \frac{a}{2}\left(\left(1+a^{-2k}\right)\left(bu_1+b^{-1}u_2\right)^k + (1-a^{-2k})\left(bu_1-b^{-1}u_2\right)^k\right)$$
$$u_2' = \frac{a}{2}\left(\left(1+a^{-2k}\right)\left(bu_3+b^{-1}u_4\right)^k + (1-a^{-2k})\left(bu_3-b^{-1}u_4\right)^k\right)$$
$$u_3' = \frac{a}{2}\left(\left(1+a^{-2k}\right)\left(b^{-1}u_1+bu_2\right)^k + \left(1-a^{-2k}\right)\left(b^{-1}u_1-bu_2\right)^k\right)$$
$$u_4' = \frac{a}{2}\left(\left(1+a^{-2k}\right)\left(b^{-1}u_3+bu_4\right)^k + (1-a^{-2k})\left(b^{-1}u_3-bu_4\right)^k\right).$$

The following choice of reduced variables is convenient for analysis on phase diagram. Note that the paramagnetic phase is a high symmetry phase (we will have $u_1 = u_4$ and $u_2 = u_3$). Thus, for analysis on phase diagram, we will reduce our variables by the following form:

$$x = \frac{u_2+u_3}{u_1+u_4},\ y_1 = \frac{u_1-u_4}{u_1+u_4},\ y_2 = \frac{u_2-u_3}{u_1+u_4} \quad . \tag{2}$$

Finally the general systems for even order now have the following form:

$$x' = \frac{1}{D}[\left(1+a^{-2k}\right)\left[\left(b\left(x-y_2\right)+b^{-1}\left(1-y_1\right)\right)^k + \left(b^{-1}\left(1+y_1\right)+b\left(x+y_2\right)\right)^k\right]$$
$$+\left(1-a^{-2k}\right)\left[\left(b\left(x-y_2\right)-b^{-1}\left(1-y_1\right)\right)^k + \left(b^{-1}\left(1+y_1\right)-b\left(x+y_2\right)\right)^k\right]]$$
$$y_1' = \frac{1}{D}[\left(1+a^{-2k}\right)\left[\left(b\left(1+y_1\right)+b^{-1}\left(x+y_2\right)\right)^k - \left(b^{-1}\left(x-y_2\right)+b\left(1-y_1\right)\right)^k\right]$$
$$+\left(1-a^{-2k}\right)\left[\left(b\left(1+y_1\right)-b^{-1}\left(x+y_2\right)\right)^k - \left(b^{-1}\left(x-y_2\right)-b\left(1-y_1\right)\right)^k\right]]$$
$$y_2' = \frac{1}{D}[\left(1+a^{-2k}\right)\left[\left(b\left(x-y_2\right)+b^{-1}\left(1-y_1\right)\right)^k - \left(b^{-1}\left(1+y_1\right)+b\left(x+y_2\right)\right)^k\right]$$

and
$$+\left(1-a^{-2k}\right)\left[\left(b\left(x-y_2\right)-b^{-1}\left(1-y_1\right)\right)^k - \left(b^{-1}\left(1+y_1\right)-b\left(x+y_2\right)\right)^k\right]] \tag{3}$$
$$D = \left(1+a^{-2k}\right)\left[\left(b\left(1+y_1\right)+b^{-1}\left(x+y_2\right)\right)^k + \left(b^{-1}\left(x-y_2\right)+b\left(1-y_1\right)\right)^k\right]$$
$$+\left(1-a^{-2k}\right)\left[\left(b\left(1+y_1\right)-b^{-1}\left(x+y_2\right)\right)^k + \left(b^{-1}\left(x-y_2\right)-b\left(1-y_1\right)\right)^k\right] \quad .$$

B. Odd k

The 4 independent variables are as follows:

$$z_1 = Z^{(n)}\begin{pmatrix}++\dots+\\ +\end{pmatrix} = a^k\left[Z^{(n)}(+,+)\right]^k$$
$$z_2 = Z^{(n)}\begin{pmatrix}--\dots-\\ +\end{pmatrix} = a^{-k}\left[Z^{(n)}(+,-)\right]^k$$

$$z_3 = Z^{(n)}\begin{pmatrix} ++\ldots+ \\ - \end{pmatrix} = a^k\left[Z^{(n)}(-,+)\right]^k$$

$$z_4 = Z^{(n)}\begin{pmatrix} --\ldots- \\ - \end{pmatrix} = a^{-k}\left[Z^{(n)}(-,-)\right]^k.$$

When the number of spins down m is even, then,

$$Z^{(n)}\begin{pmatrix} \sigma(y_1),\sigma(y_2),\ldots,\sigma(y_k) \\ + \end{pmatrix} = a^{2m} z_1^{\frac{k-m}{k}} z_2^{\frac{m}{k}},$$

$$Z^{(n)}\begin{pmatrix} \sigma(y_1),\sigma(y_2),\ldots,\sigma(y_k) \\ - \end{pmatrix} = a^{2m} z_3^{\frac{k-m}{k}} z_4^{\frac{m}{k}}.$$

And when the number of spins down m is odd, then,

$$Z^{(n)}\begin{pmatrix} \sigma(y_1),\sigma(y_2),\ldots,\sigma(y_k) \\ + \end{pmatrix} = a^{-2(k-m)} z_1^{\frac{k-m}{k}} z_2^{\frac{m}{k}},$$

$$Z^{(n)}\begin{pmatrix} \sigma(y_1),\sigma(y_2),\ldots,\sigma(y_k) \\ - \end{pmatrix} = a^{-2(k-m)} z_3^{\frac{k-m}{k}} z_4^{\frac{m}{k}}.$$

Now we introduced a new variable u_i with $u_i = \sqrt[k]{z_i}$; $z_i = u_i^k$. Then the recurrence equations are reduced to these recurrent systems:

$$u_1' = \frac{a}{2}\left(\left(1+a^{-2k}\right)\left(bu_1 + a^2b^{-1}u_2\right)^k + (1-a^{-2k})\left(bu_1 - a^2b^{-1}u_2\right)^k\right)$$

$$u_2' = \frac{a^{-1}}{2}\left(\left(1+a^{-2k}\right)\left(bu_3 + a^2b^{-1}u_4\right)^k + (1-a^{-2k})\left(bu_3 - a^2b^{-1}u_4\right)^k\right)$$

$$u_3' = \frac{a}{2}\left(\left(1+a^{-2k}\right)\left(b^{-1}u_1 + a^2bu_2\right)^k + \left(1-a^{-2k}\right)\left(b^{-1}u_1 - a^2bu_2\right)^k\right)$$

$$u_4' = \frac{a^{-1}}{2}\left(\left(1+a^{-2k}\right)\left(b^{-1}u_3 + a^2bu_4\right)^k + (1-a^{-2k})\left(b^{-1}u_3 - a^2bu_4\right)^k\right).$$

Using the same choice of reduced variables (2) for x, y_1 and y_2 as in even order, finally we will have following recurrent equations:

$$x' = \frac{1}{D}[a^{-1}\left[\left(1+a^{-2k}\right)\left(b(x-y_2)+a^2b^{-1}(1-y_1)\right)^k + \left(1-a^{-2k}\right)\left(b(x-y_2)-a^2b^{-1}(1-y_1)\right)^k\right]$$
$$+a\left[\left(1+a^{-2k}\right)\left(b^{-1}(1+y_1)+a^2b(x+y_2)\right)^k + \left(1-a^{-2k}\right)\left(b^{-1}(1+y_1)-a^2b(x+y_2)\right)^k\right]]$$

$$y_1' = \frac{1}{D}[a\left[\left(1+a^{-2k}\right)\left(b(1+y_1)+a^2b^{-1}(x+y_2)\right)^k + \left(1-a^{-2k}\right)\left(b(1+y_1)-a^2b^{-1}(x+y_2)\right)^k\right]$$
$$-a^{-1}\left[\left(1+a^{-2k}\right)\left(b^{-1}(x-y_2)+a^2b(1-y_1)\right)^k + \left(1-a^{-2k}\right)\left(b^{-1}(x-y_2)-a^2b(1-y_1)\right)^k\right]]$$

$$y_2' = \frac{1}{D}[a^{-1}\left[\left(1+a^{-2k}\right)\left(b(x-y_2)+a^2b^{-1}(1-y_1)\right)^k + \left(1-a^{-2k}\right)\left(b(x-y_2)-a^2b^{-1}(1-y_1)\right)^k\right]$$
$$-a\left[\left(1+a^{-2k}\right)\left(b^{-1}(1+y_1)+a^2b(x+y_2)\right)^k + \left(1-a^{-2k}\right)\left(b^{-1}(1+y_1)-a^2b(x+y_2)\right)^k\right]]$$

and (4)

$$D = a\left[\left(1+a^{-2k}\right)\left(b(1+y_1)+a^2b^{-1}(x+y_2)\right)^k + \left(1-a^{-2k}\right)\left(b(1+y_1)-a^2b^{-1}(x+y_2)\right)^k\right]$$
$$+a^{-1}\left[\left(1+a^{-2k}\right)\left(b^{-1}(x-y_2)+a^2b(1-y_1)\right)^k + \left(1-a^{-2k}\right)\left(b^{-1}(x-y_2)-a^2b(1-y_1)\right)^k\right].$$

Remark Note that these produced recurrent equations (3) and (4) are essentially different for even and odd *k*. For even k we have fixed point $(x^*, 0, 0)$ but there is no fixed point for odd *k*.

II. Resultant Phase Diagram:

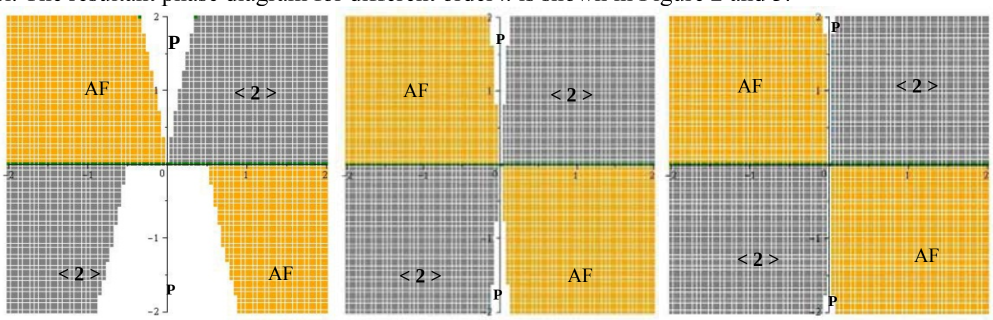
The result of analysis on recurrent systems (3) and (4) can be summarized through phase diagram. The phase diagram will be produced after a numerical calculation.

Starting from initial conditions, the recurrent systems (3) and (4) is iterated for a large number of iteration; i.e. 10000 times. The behaviour around ten last iteration will determined the type of phase for each parameter α and β.

Let $\alpha = \frac{T}{J_0}$ be a vertical axis and $\beta = -\frac{J_p}{J_0}$ be a horizontal axis in the diagrams.

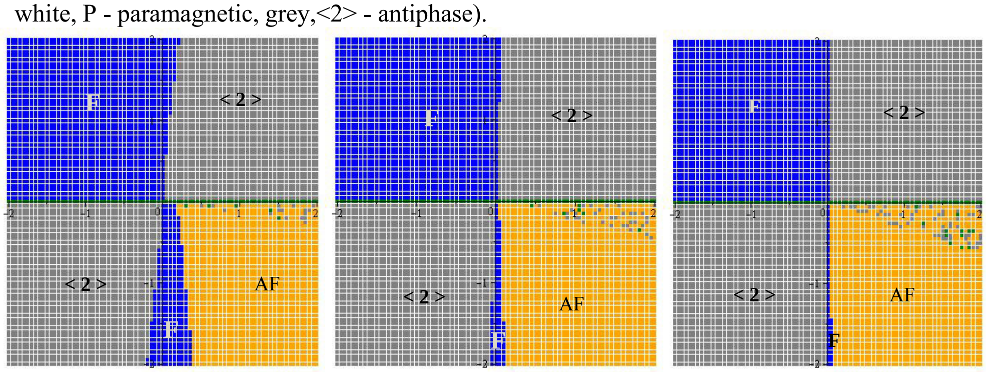
When the behaviour reached a fixed point (x^*, y_1^*, y_2^*), this correspond to paramagnetic phase if $y_1^* = y_2^* = 0$ or to ferromagnetic phase if $y_1^* = y_2^* \neq 0$. Next the system may become periodic with period p where $p = 2$ means antiferromagnetic phase and for $p = 4$ are called antiphase, denoted as <2> for compactness. Finally the system may remain aperiodic.

It is difficult to differentiate numerically behaviour between a truly aperiodic and one with very long period. Determine the character for each point, we get a diagram describing the existence of magnetic phase in the model. The resultant phase diagram for different order *k* is shown in Figure 2 and 3.



(a) (b) (c)

Fig. 2: Phase diagram for even order *k* (a) order 2, (b) order 4 and (c) order 6 (orange, AF - antiferromagnetic, white, P - paramagnetic, grey,<2> - antiphase).



(a) (b) (c)

Fig. 3: Phase diagram for odd order *k* (a) order 3, (b) order 5 and (c) order 7 (blue, F - ferromagnetic, orange, AF - antiferromagnetic, grey, <2> - antiphase).

We show phase diagram of even and odd order 2, 3, 4, 5, 6, and 7 as in Figure 2 and 3 for some initial conditions. Fixing the boundary condition $\overline{\sigma}_{(n)} = 1$, the initial condition for even *k* is as below:

$$x^{(1)} = 1, \quad y_1^{(1)} = \frac{b^2 - 1}{b^2 + 1}, \quad y_2^{(1)} = \frac{b^2 - 1}{b^2 + 1} \tag{5}$$

And for odd order *k* is

$$x^{(1)} = \frac{b^2 + a^2}{a^2 b^2 + 1}, \quad y_1^{(1)} = \frac{a^2 b^2 - 1}{a^2 b^2 + 1}, \quad y_2^{(1)} = \frac{b^2 - a^2}{a^2 b^2 + 1} \quad . \tag{6}$$

Figure 2 shows the resultant phase diagram of k = {2, 4, 6} for initial condition (5). The phase exist for this order are antiferromagnetic or period 2 phase(orange), paramagnetic phase(white) and antiphase-<2> or period 4 phase(grey). The region for both left and right hand side are symmetrical to each other (in paramagnetic phase region).

From left, at first and second quadrant the phases existed are antiferro → para → antiphase. Inversely at third and fourth quadrant, the phases are antiphase → para → antiferro. However the size of paramagnetic phase region is decreasing as the order k increases and remains the increasing size of antiferromegnetic and antiphase phase regions.

Figure 3 shows the resultant phase diagram of k = {3, 5, 7} for initial condition (6). The phase exist for this order are ferromagnetic phase(blue), antiferromagnetic or period 2 phase(orange), and antiphase-<2> or period 4 phase(grey).

From left, at first and second quadrant we have ferro → antiphase and at third and fourth quadrant we have antiphase → ferro → antiferro. In contrast with phase diagram of even order, no paramagnetic phase is found in any odd order.

Furthermore, in the second quadrant, there are a duality condition as mentioned in (Nasir Ganikhodjaev, Siti Fatimah Zakaria, 2011 specifically for $k = 2$. For random initial conditions two types of phases can be found in the second quadrant either independently exist or combination of both types of phases. We found that there are weak stability in the second quadrant for this model at $k = 2$ when there are no nearest-neighbour interactions.

III. Paramagnetic Transition Analysis:

An analytical method will be used to analyze the transition line of para-antifero or para-<2> as shown in a phase diagram.

Since a standard fixed point $(x^*, 0, 0)$ can be found for even order k, it is convenient to analyze the stability of paramagnetic phase and obtained the transition line by linearization and approximation approach.

However the case of odd order k will be left for future. This corresponds to the non-existence of paramagnetic phase in the diagram. Currently, a suitable approach to analyze a non-standard fixed point in odd order such that $y_1 = y_2 \neq 0$ is still in doubt.

Thus, the analysis on paramagnetic phase in this section will cover only the even order k. At (3), let $y_1 = y_2 = 0$ we get a standard fixed point equation for x. Let $x = x^*$, then a general fixed point for even k x^* is given by

$$x^* = \frac{(1+a^{-2k})(b^2 x^* + 1)^k + (1-a^{-2k})(b^2 x^* - 1)^k}{(1+a^{-2k})(b^2 + x^*)^k + (1-a^{-2k})(b^2 - x^*)^k} \quad .$$

Linearizing the system of general even order using Jacobian, we have in matrix form

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where

$$J_{11} = \frac{kb(1+a^{-2k})(b+xb^{-1})^{k-1} + kb(1-a^{-2k})(b-xb^{-1})^{k-1}}{(1+a^{-2k})(b+xb^{-1})^k + (1-a^{-2k})(b-xb^{-1})^k}$$

$$J_{12} = \frac{kb^{-1}(1+a^{-2k})(b+xb^{-1})^{k-1} + kb^{-1}(1-a^{-2k})(b-xb^{-1})^{k-1}}{(1+a^{-2k})(b+xb^{-1})^k + (1-a^{-2k})(b-xb^{-1})^k}$$

$$J_{21} = \frac{-kb^{k-1}x(1+a^{-2k})(b+xb^{-1})^{k-1} + kb^{k-1}x(1-a^{-2k})(b-xb^{-1})^{k-1}}{(1+a^{-2k})(b^2x+1)^k + (1-a^{-2k})(b^2x-1)^k}$$

$$J_{22} = \frac{-kb^{k+1}x(1+a^{-2k})(b+xb^{-1})^{k-1} + kb^{k+1}x(1-a^{-2k})(b-xb^{-1})^{k-1}}{(1+a^{-2k})(b^2x+1)^k + (1-a^{-2k})(b^2x-1)^k}$$

Put into account its eigenvalue λ, the final secular equation is as follows:

$$\lambda^2 - (J_{11} + J_{22})\lambda + (J_{11}J_{22} - J_{21}J_{12}) = 0 \quad .$$

The fixed point is linearly stable if eigenvalue have moduli smaller than 1; $|\lambda| < 1$.

Para - <2> Transition

Since the transition in first and second quadrant is symmetrical to each other, it is enough to check only one quadrant (either first or second). The transition in third and fourth quadrant may be unstable and more complicated. So this will be left for future investigation.

Specific case of even order will be discussed. The case of order $k = 2$ has been describe in (Ganikhodjaev N., Siti Fatimah Zakaria, 2011). In this subsection, the first quadrant $(J_p < 0, J_0 > 0)$ with order $k = 4$ will be our main focus. For para-<2> transition, the eigenvalue λ is a complex variable since the fixed point is approached in an oscillatory way. Substituting $k = 4$ and $\lambda = i$, the secular equation is reduced to

$$4b^2a^{16}x^8 + 4x^7(7b^4a^8 - b^{10}a^8) + x^6(48b^6 - 16b^8 + 36b^6a^{16} - 12b^{12}a^{16}) + x^5(140b^8a^8 - 12b^{14}a^8 - 48b^{10}a^8 - 24b^6a^8) + x^4(64b^{10} - 16b^4 - 48b^{12} + 76b^{10}a^{16} - 72b^{10}a^8 - 4b^{16}a^{16}) + x^3(84b^{12}a^8 - 16b^{14}a^8 - 72b^{10}a^8 - 48b^6a^8 - 4b^2a^8) + x^2(16b^{14} - 48b^8 + 12b^{14}a^{16} - 24b^{12}a^{16} - 12b^4a^{16}) + x(4b^{16}a^8 - 16b^{10}a^8 - 12b^6a^8) - 4b^8a^{16} = 0.$$

At low temperature $T \sim 0$, the parameter $b \sim 0, c \sim \infty$ and for some conditions $bc \sim \infty$. Thus the positive x^* can be approximated as $x^* \sim \frac{1}{3b^6}$. Substituting x^*, the equality $b^{14} = \frac{1}{3}$ or $\beta = \frac{\alpha}{14}\ln 3$ corresponds to the asymptote of the transition line in the first quadrant. This is in agreement with the phase diagram obtained numerically.

Symmetrically, the inverse is corresponds to asymptote of transition line in the second quadrant$\left(\beta = -\frac{\alpha}{14}\ln 3\right)$. See **Fig. 4:**

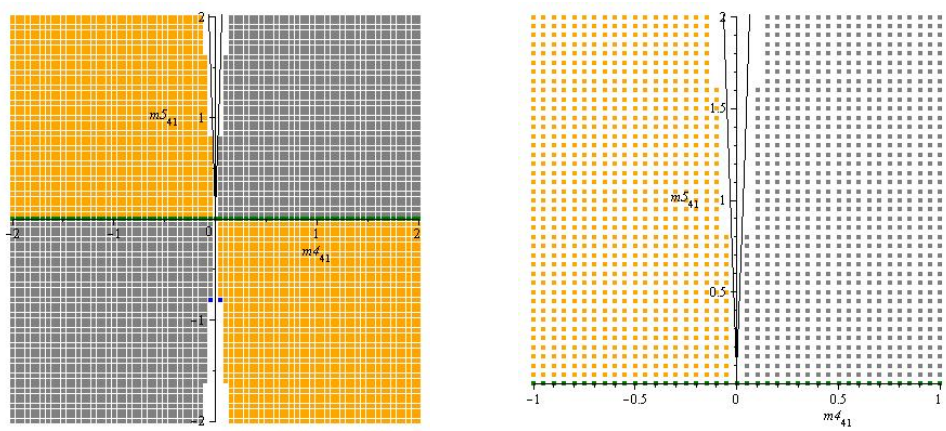


Fig. 4: (a) Phase diagram of order 4 with asymptote of transition line. (b) Enlargement - first and second quadrant phase diagram of order 4.

For $k = 6$, we use the same way to produce asymptote of transition line that is in agreement with numerical diagram. Similarly for higher order but only need more calculation.

Conclusion:

In this paper, the phase diagram is produced numerically for different order; for k is even and k is odd by using derived general recurrent system of equations. Then the general fixed point equation is produced for even order. Later the system of equations been analyze around fixed point for the case of $k = 4$.

Phase diagram between even and odd order can be distinguish by looking at the existence of paramagnetic phase. The paramagnetic phase exists only in diagram of even order, however the size of region keep on decreases as order k increases. Moreover, the analytical analysis on recurrent relation of order 4 has been proved to be in agreement with one obtained numerically.

However, the odd order analysis become more complicated since there is no fixed point for odd order recurrent equation. The study on third and fourth quadrant for even order may be also complicated to solve analytically. Thus both questions will be left for future interest.

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